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# Permanence and global stability for nonautonomous $N$-species Lotka–Volterra competitive system with impulses and infinite delays

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ABSTRACT

In this paper, we consider a class of nonautonomous $N$-species Lotka–Volterra competitive systems with impulses and infinite delays. By developing the methods given in Teng (2002) [25], we give sufficient conditions for permanence and global attractivity of the system.



## 1. Introduction

As we well know, in the theory of mathematical biology, traditional Lotka–Volterra competition systems with delays or without delays are very important mathematical models which describe multi-species population dynamics in a nonautonomous environment. Many important and interesting results on the dynamical behaviors for such systems, such as the permanence, extinction, global asymptotic behavior and the existence and uniqueness of coexistence states (for example, positive periodic solution, positive almost periodic solution, etc.), can be found in [1–4,13,14,19,24–30] and references therein.

However, owing to many natural and man-made factors, such as fire, drought, flooding, crop-dusting, deforestation, hunting, harvesting, etc., the intrinsic discipline of biological species or ecological environment usually undergoes some discrete changes of relatively short duration at some fixed times, which makes it unsuitable to be considered continually. For having a more accurate description of such system, we need to consider the impulsive differential equations. With the development of the theory of impulsive differential equations (the fundamental theory of impulsive differential equations can be seen in the monographs [9,18,21,23]), we can establish adequate mathematical models of impulsive differential equations to investigate the dynamic behaviors of such ecosystems with impulsive effects.

Recently, the dynamical behaviors of impulsive population dynamical systems have been extensively studied. Many important and interesting population dynamical systems have been extensively studied, see [5–8,10,15–17,22,31]. In [6], the authors investigated the following nonautonomous $N$-species Lotka–Volterra competitive system with impulsive effects

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$$\dot{u}_i(t) = u_i(t)\left(a_i(t) - \sum_{l=1}^{n} b_{il}(t)u_j(t)\right), \quad t \neq t_k,$$
$$u_i\left(t_k^+\right) = (1 + p_{ik})u_i(t_k), \quad i = 1, 2, \ldots, n, \; k = 1, 2, \ldots.$$

Some criteria on the persistence of species and global stability of the system are established by means of the average value conditions and the methods of inequalities estimate and Lyapunov functions. Hou and Teng in [15] investigated the following nonautonomous $N$-species Lotka–Volterra competitive system with impulses

$$\dot{x}_i(t) = x_i(t)\left(a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t)\right), \quad t \neq t_k,$$
$$x_i\left(t_k^+\right) = h_{ik}x_i(t_k), \quad i = 1, 2, \ldots, n, \; k = 1, 2, \ldots.$$

The sufficient conditions on the permanence and global attractivity are established by the inequalities estimate and Lyapunov functions and the corresponding results given in [6] are improved and extended. Motivated by above works, in this paper, we will study the following nonautonomous $N$-species Lotka–Volterra competitive system with impulses and infinite delays:

$$\dot{x}_i(t) = x_i(t)\left(a_i(t) - b_{ii}(t)x_i(t) - \sum_{j=1,\, j\neq i}^{n} b_{ij}(t) \int_{-\infty}^{0} k_i(s)x_j(t+s)\,\mathrm{d}s\right), \quad t \neq t_k,$$
$$x_i\left(t_k^+\right) = h_{ik}x_i(t_k), \quad k = 1, 2, \ldots, \; i = 1, 2, \ldots, n, \tag{1.1}$$

where $a_i(t)$ and $b_{ij}(t)$ are defined on $[0, +\infty)$ and are bounded and continuous functions, $b_{ij}(t) \geqslant 0$ for all $i, j = 1, 2, \ldots, n$, $t \in [0, +\infty)$, $k_i(s)$ is defined on $(-\infty, 0]$, and is a nonnegative and integrable function, $\int_{-\infty}^{0} k_i(s)\,\mathrm{d}s = 1$ for $i = 1, 2, \ldots, n$ and impulsive coefficients $h_{ik}$ are positive constants for any $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots$. We will establish some new sufficient conditions on the permanence of species and global attractivity for system (1.1). We will see that in many special cases these conditions can be easily checked.

The organization of this paper is as follows. In the next section we will introduce the main lemmas. In Section 3, conditions for the permanence of the system are considered. In Section 4, we establish conditions for global attractivity of the system. In the last section, a suitable example is given to illustrate that our main results are applicable.

## 2. Preliminaries

Let $R_+ = [0, \infty)$, $R_- = (-\infty, 0]$, $R_+^n = \{x \in R^n\colon\ x_i \geqslant 0,\ i = 1, 2, \ldots, n\}$ and $\operatorname{int} R_+^n = \{x \in R^n\colon\ x_i > 0,\ i = 1, 2, \ldots, n\}$. Let $\{t_k\}$ be a time sequence and $t^* > -\infty$, satisfying $t^* < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$ as $k \to \infty$. Let $I \subset R$ be any interval. Define $PC(I, R^n) = \{\phi : I \to R^n \mid \phi$ is continuous everywhere except at the points $t = t_k \in I$ and $\phi(t_k^-)$ and $\phi(t_k^+)$ exist with $\phi(t_k^-) = \phi(t_k)\}$. Define $PCB = \{\phi \in PC(R_-, R^n)\colon\ \phi$ is bounded$\}$. For any $\phi \in PCB$ the norm of $\phi$ is defined by $\|\phi\| = \sup_{\theta \in R_-} |\phi(\theta)|$. Let set $PCB_+ = \{\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in PCB\colon\ \phi_i(\theta) \geqslant 0$ for all $\theta \in R_-$ and $\phi_i(0^+) > 0$ for $i = 1, 2, \ldots, n\}$.

Motivated by the biological background of system (1.1), in this paper, we always assume that all solutions of system (1.1) satisfy the following initial conditions

$$x_i(s, \phi) = \phi_i(s), \quad s \in R_-,$$
$$x_i\left(0^+, \phi\right) = \phi_i\left(0^+\right), \quad s \in R_-, \tag{2.1}$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in PCB_+$. By the fundamental theory of functional differential equation with impulsive (see [12]), system (1.1) has a unique solution $x(t, \phi)$ satisfying the initial condition (2.1), where $x(t, \phi) = (x_1(t, \phi), \ldots, x_n(t, \phi))$. It is obvious that the solution $x(t, \phi)$ is positive, i.e., $x_i(t, \phi) > 0$ $(i = 1, 2, \ldots, n)$ in its maximal interval of the existence, denoted by $J^+ = J^+(\phi)$.

Note that the solution $x(t) = x(t, \phi)$ of problems (1.1) and (2.1) is piecewise continuous function in the interval $J^+(\phi)$ with points of discontinuity of the first kind at $t_k$ $(k = 1, 2, \ldots)$ at which is left continuous, i.e., the following relations are satisfied:

$$x_i\left(t_k^-\right) = x_i(t_k), \quad k = 1, 2, \ldots,$$
$$x_i\left(t_k^+\right) = h_{ik}x_i(t_k), \quad t_k \in J^+(\phi), \; i = 1, 2, \ldots, n.$$

Now, we introduce several lemmas which will be very useful in the proofs of the main results.

We first consider the following impulsive logistic system

$$\dot{x}(t) = x(t)\left(\alpha(t) - \beta(t)x(t)\right), \quad t \neq t_k,$$
$$x\left(t_k^+\right) = h_k x(t_k), \quad k = 1, 2, \ldots, \tag{2.2}$$

where $\alpha(t)$ and $\beta(t)$ are bounded and continuous functions defined on $R_+$, $\beta(t) \geqslant 0$ for all $t \in R_+$ and impulsive coefficients $h_k$ for any $k = 1, 2, \ldots$ are positive constants. We have the following result.

**Lemma 2.1.** *Suppose that there is a positive constant $\omega$ such that*

$$\liminf_{t\to\infty} \int_t^{t+\omega} \beta(s)\,ds > 0, \qquad \liminf_{t\to\infty}\left(\int_t^{t+\omega} \alpha(s)\,ds + \sum_{t\leqslant t_k<t+\omega} \ln h_k\right) > 0,$$

*and function*

$$h(t,\nu) = \sum_{t\leqslant t_k<t+\nu} \ln h_k$$

*is bounded on $t \in R_+$ and $\nu \in [0,\omega)$. Then we have*:

(a) *There exist positive constants $m$ and $M$ such that*

$$m \leqslant \liminf_{t\to\infty} x(t) \leqslant \limsup_{t\to\infty} x(t) \leqslant M$$

*for any positive solution $x(t)$ of system* (2.2).

(b) $\lim_{t\to\infty}(x(t) - y(t)) = 0$ *for any two positive solutions $x(t)$ and $y(t)$ of system* (2.2).

**Remark 2.1.** When system (2.2) is periodic, that is, we can assume that there is a positive constant $\omega$ and a positive integer $q$ such that $\alpha(t+\omega) = \alpha(t)$, $\beta(t+\omega) = \beta(t)$, $t_{k+q} = t_k + \omega$ and $h_{k+q} = h_k$ for all $t \in R_+$ and $k = 1, 2, \ldots$, then the conditions in Lemma 2.1 are equivalent to

$$\int_0^{\omega} \beta(s)\,ds > 0 \quad \text{and} \quad \int_0^{\omega} \alpha(s)\,ds + \sum_{k=1}^{q} \ln h_k > 0. \tag{2.3}$$

In addition, function $h(t,\mu)$ satisfies

$$\left|h(t,\nu)\right| = \left|\sum_{t\leqslant t_k<t+\nu} \ln h_k\right| \leqslant \sum_{k=1}^{q} |\ln h_k| \quad \text{for all } t \in R_+ \text{ and } \nu \in [0,\omega).$$

Therefore, if condition (2.3) holds for the periodic system (2.2), then conclusions (a) and (b) of Lemma 2.1 also hold.

**Remark 2.2.** In [20], the authors studied periodic system (2.2) in detail and obtained that if condition (2.3) holds, then periodic system (2.2) has a unique positive $\omega$-periodic solution $x^*(t)$ which is globally asymptotically stable. See Theorem 2.1 in [20].

We can find above lemma in [15].

Let $f(s)$ be a continuous and bounded function defined on $R_+$, set $f^L = \inf\{f(s)\colon s \in R_+\}$ and $f^M = \sup\{f(s)\colon s \in R_+\}$. We define functions

$$\eta(t) = 1 - e^{-\alpha^M(t-t_k)} + \sum_{i=1}^{k-1}\left(e^{-\alpha^M(t-t_{i+1})} - e^{-\alpha^M(t-t_i)}\right) \prod_{j=i+1}^{k} h_j^{-1},$$

$$\xi(t) = 1 - e^{-\alpha^L(t-t_k)} + \sum_{i=1}^{k-1}\left(e^{-\alpha^L(t-t_{i+1})} - e^{-\alpha^L(t-t_i)}\right) \prod_{j=i+1}^{k} h_j^{-1},$$

$$\rho(t) = e^{-\alpha^L(t-t_1)} \prod_{i=1}^{k} h_i^{-1} \quad \text{and} \quad \zeta(t) = e^{-\alpha^M(t-t_1)} \prod_{i=1}^{k} h_i^{-1},$$

for all $t \in R_+$ and $t \in [t_k, t_{k+1})$, $k = 1, 2, \ldots$. Then we have the following result for system (2.2).

**Lemma 2.2.** *Suppose $\alpha^L, \beta^L > 0$, $\lim_{t\to\infty} \rho(t) = 0$ and there exist positive constants $\underline{\eta}$ and $\overline{\xi}$ such that*

$$\liminf_{t\to\infty} \eta(t) \geqslant \underline{\eta} \quad \text{and} \quad \limsup_{t\to\infty} \xi(t) \leqslant \overline{\xi}.$$

*Then we have*:

(a) *System* (2.2) *is permanent and*

$$\frac{\alpha^L}{\beta^M\overline{\xi}} \leqslant \liminf_{t\to\infty} x(t) \leqslant \limsup_{t\to\infty} x(t) \leqslant \frac{\alpha^M}{\beta^L\underline{\eta}}.$$

(b) $\lim_{t\to\infty}(x(t)-y(t))=0$ *for any two positive solutions* $x(t)$ *and* $y(t)$ *of system* (2.2).

**Proof.** Firstly, we prove

$$\limsup_{t\to\infty} x(t) \leqslant \frac{\alpha^M}{\beta^L\underline{\eta}}. \tag{2.4}$$

Let $z(t)=\frac{1}{x(t)}$, then we have

$$z(t)=\beta(t)-\alpha(t)z(t), \quad t\neq t_k,$$
$$z\big(t_k^+\big)=h_k^{-1}z(t_k), \quad k=1,2,\ldots.$$

By the variation of constant formula, we obtain

$$z(t)=\int_{t_k}^{t}\beta(s)\exp\left(-\int_s^t \alpha(u)\,\mathrm{d}u\right)\mathrm{d}s+h_k^{-1}z(t_k)\exp\left(-\int_{t_k}^t \alpha(s)\,\mathrm{d}s\right),$$

for $t\in[t_k,t_{k+1})$.

From above equality, we can get

$$z(t)\geqslant\frac{\beta^L}{\alpha^M}\big(1-e^{-\alpha^M(t-t_k)}\big)+z(t_k)h_k^{-1}e^{-\alpha^M(t-t_k)}.$$

Then we have

$$x(t)\leqslant\frac{h_kx(t_k)}{h_kx(t_k)\frac{\beta^L}{\alpha^M}(1-e^{-\alpha^M(t-t_k)})+e^{-\alpha^M(t-t_k)}}. \tag{2.5}$$

Consequently,

$$x(t_{k+1})\leqslant\frac{h_kx(t_k)}{h_kx(t_k)\frac{\beta^L}{\alpha^M}(1-e^{-\alpha^M(t_{k+1}-t_k)})+e^{-\alpha^M(t_{k+1}-t_k)}}.$$

By the iteration, we obtain

$$x(t_{k+1})\leqslant\left(\frac{\beta^L}{\alpha^M}\sum_{i=1}^{k}\big(e^{-\alpha^M(t_{k+1}-t_{i+1})}-e^{-\alpha^M(t_{k+1}-t_i)}\big)\prod_{j=i+1}^{k}h_j^{-1}+e^{-\alpha^M(t_{k+1}-t_1)}x(t_1)^{-1}\prod_{i=1}^{k}h_i^{-1}\right)^{-1}, \tag{2.6}$$

where $\prod_{i=k+1}^{k}h_i^{-1}=1$. Submitted inequality (2.6) into (2.5), we get that

$$x(t)\leqslant\left(\frac{\beta^L}{\alpha^M}\eta(t)+\frac{\zeta(t)}{x(t_1)}\right)^{-1}.$$

From $\lim_{t\to\infty}\rho(t)=0$, we have $\lim_{t\to\infty}\zeta(t)=0$. Therefore, from the assumption of the lemma we have (2.4) holds.

By the similar argument as in the proof of (2.4) we can prove that

$$\liminf_{t\to\infty} x(t)\geqslant\frac{\alpha^L}{\beta^M\overline{\xi}}.$$

Conclusion (a) is proved.

By the similar argument as in the proof of (b) of Lemma 2.1 in [15], we have conclusion (b) holds. This completes the proof of the lemma. □

**Remark 2.3.** For a system (2.2) without impulsive effect, i.e. $h_k=1$ for all $k=1,2,\ldots$, we have $\eta(t)=1-e^{-a^M(t-t_1)}$, $\xi(t)=1-e^{-a^L(t-t_1)}$ and $\zeta(t)=e^{-\alpha^M(t-t_1)}$. Therefore, we can obtain that $\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}\xi(t)=1$ and

$$\frac{\alpha^L}{\beta^M}\leqslant\liminf_{t\to\infty}x(t)\leqslant\limsup_{t\to\infty}x(t)\leqslant\frac{\alpha^M}{\beta^L},$$

which was given by Lemma 1.4 of Chen [11]. Therefore Lemma 2.2 is an extension of result on the single species Lotka–Volterra system.

**Remark 2.4.** When $h_k \equiv 1$ for all $k = 1, 2, \ldots$, obviously, we can obtain that the conditions of Lemma 2.1 hold if the assumptions of Lemma 2.2 hold. Therefore, there exists an interesting open question: when $h_k \not\equiv 1$ for $k = 1, 2, \ldots$ could conditions of Lemma 2.2 imply the conditions of Lemma 2.1?

Next, we introduce an important comparison theorem of impulsive differential equation [18].

**Lemma 2.3.** *Assume that $m \in PC(R_+, R)$ with points of discontinuity at $t = t_k$ and is left continuous at $t = t_k$, $k = 1, 2, \ldots$, and that*

$$\begin{cases} Dm(t) \leqslant g\big(t, m(t)\big), & t \neq t_k,\ k = 1, 2, \ldots, \\ m\big(t_k^+\big) \leqslant \phi_k\big(m(t_k)\big), & t = t_k,\ k = 1, 2, \ldots, \end{cases} \tag{2.7}$$

*where $g \in C(R_+ \times R_+, R)$, $\phi_k \in C(R, R)$ and $\phi_k(u)$ is nondecreasing in $u$ for each $k = 1, 2, \ldots$. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation*

$$\begin{cases} \dot{u}(t) = g\big(t, u(t)\big), & t \neq t_k,\ k = 1, 2, \ldots, \\ m\big(t_k^+\big) = \phi_k\big(u(t_k)\big), & t = t_k,\ k = 1, 2, \ldots, \\ u\big(t_0^+\big) = u_0, \end{cases} \tag{2.8}$$

*existing on $[t_0, \infty)$; then $m(t_0^+) \leqslant u_0$ implies $m(t) \leqslant r(t)$, $t \geqslant t_0$.*

**Remark 2.5.** In Lemma 2.3, assume that inequalities (2.7) are reversed. Let $p(t)$ be the minimal solution of (2.8) existing on $(t_0, +\infty)$. Then, $m(t_0^+) \geqslant u_0$ implies $m(t) \geqslant p(t)$, $t \geqslant t_0$.

The following lemma will be used in the proof of the global attractivity of system (1.1).

**Lemma 2.4.** *Let function $f(t)$ be continuous and the right upper Dini derivative exist on $[0, +\infty)$, and there exist a sequence $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ and a positive constant $M$, such that $\frac{\mathrm{d}f(t)}{\mathrm{d}t}$ exist for all $t \in R_+$ and $t \neq t_k$ ($k \in N$) and $|D^+ f(t)| < M$. Then $f(t)$ is uniformly continuous on $[0, +\infty)$.*

**Proof.** From the assumption of the lemma, for any $s_1, s_2 \in [0, \infty)$, we have

$$\big|f(s_1) - f(s_2)\big| = \left| \int_{s_1}^{s_2} D^+ f(s)\, \mathrm{d}s \right| \leqslant M|s_1 - s_2|.$$

Therefore, the conclusion of the lemma holds. $\square$

## 3. Permanence

In this section we will study the permanence of all species of system (1.1). For each $i = 1, 2, \ldots, n$, we consider the following impulsive logistic systems as the subsystems of system (1.1)

$$\begin{aligned} &\dot{x}_i(t) = x_i(t)\big(a_i(t) - b_{ii}(t)x_i(t)\big), \quad t \neq t_k, \\ &x_i\big(t_k^+\big) = h_{ik}x_i(t_k), \quad k = 1, 2, \ldots. \end{aligned} \tag{3.1}$$

Assume that $x_i^*(t)$ is some fixed positive solution of system (3.1). On the permanence of all species $x_i$ ($i = 1, 2, \ldots, n$) for system (1.1) we have the following result.

**Theorem 3.1.** *Suppose that there exist constants $\omega_i$ ($i = 1, 2, \ldots, n$) such that for each $i = 1, 2, \ldots, n$*

$$\liminf_{t\to\infty} \left( \int_t^{t+\omega_i} b_{ii}(s)\, \mathrm{d}s \right) > 0, \tag{3.2}$$

$$\liminf_{t\to\infty} \int_t^{t+\omega_i} \left( a_i(s) - \sum_{j=1,\, j\neq i}^{n} b_{ij}(s)\mu_{ij}(s) \right) \mathrm{d}s + \sum_{t \leqslant t_k < t+\omega_i} \ln h_{ik} > 0, \tag{3.3}$$

*where $\mu_{ij}(t) = \int_{-t}^{0} k_i(s)x_j^*(t+s)\, \mathrm{d}s$, and*

$$h_i(t,\nu)=\sum_{t\leqslant t_k<t+\nu}\ln h_{ik}$$

*is bounded for all $t\in R_+$ and $\nu\in[0,\omega_i)$. Then system (1.1) is permanent, that is, there are positive constants $m_i$ and $M_i$ such that*

$$m_i\leqslant\liminf_{t\to\infty}x_i(t)\leqslant\limsup_{t\to\infty}x_i(t)\leqslant M_i,\quad \text{for all } i=1,2,\dots,n,$$

*for any positive solution $x(t)=(x_1(t),x_2(t),\dots,x_n(t))$ of system (1.1).*

**Proof.** From condition (3.3) we directly have

$$\liminf_{t\to\infty}\left(\int_t^{t+\omega_i}a_i(s)\,ds+\sum_{t\leqslant t_k<t+\omega_i}\ln h_{ik}\right)>0,\quad i=1,2,\dots,n.$$

Hence, from Lemma 2.1 we can obtain that there are positive constants $p_i$ and $P_i$ such that

$$p_i\leqslant x_i^*(t)\leqslant P_i\quad \text{for all } t\in R_+,\ i=1,2,\dots,n$$

and $x_i^*(t)$ is globally attractive for system (3.1).

Similar to the discussion of Theorem 3.1 in [15], we can obtain that condition (3.3) is independent of the choice of $x_i^*(t)$ $(i=1,2,\dots,n)$.

From conditions (3.2) and (3.3), there are constants $\varepsilon_0>0$ small enough and $T_0>0$ large enough such that

$$\int_t^{t+\omega_i}\left(a_i(s)-\sum_{j=1,\,j\neq i}^n b_{ij}(s)\mu_{ij}(s)-b_{ii}(s)\varepsilon_0-2\sum_{j=1,\,j\neq i}^n b_{ij}(s)\varepsilon_0\right)ds+\sum_{t\leqslant t_k<t+\omega_i}\ln h_{ik}>\varepsilon_0,\tag{3.4}$$

for all $t\geqslant T_0$ and $i=1,2,\dots,n$. Since functions $h_i(t,\nu)=\sum_{t\leqslant t_k<t+\nu}\ln h_{ik}$ $(i=1,2,\dots,n)$ are bounded for all $t\in R_+$ and $0\leqslant\nu<\omega_i$, for each $i=1,2,\dots,n$ there is a positive constant $H_i$ such that

$$\left|\sum_{t\leqslant t_k<t+\nu}\ln h_{ik}\right|\leqslant H_i,\tag{3.5}$$

for all $t\in R_+$, $0\leqslant\nu<\omega_i$. Let $x(t)=(x_1(t),\dots,x_n(t))$ be any positive solution of system (1.1). Since

$$\dot x_i(t)\leqslant x_i(t)\big(a_i(t)-b_{ii}(t)x_i(t)\big),\quad \text{for all } t\neq t_k,\ i=1,2,\dots,n,$$

by Lemma 2.3, we obtain

$$x_i(t)\leqslant u_i(t),\quad \text{for all } t\geqslant 0,$$

where $u_i(t)$ is a positive solution of system (3.1) with initial condition $u_i(0)=x_i(0)$. From Lemma 2.1, we can obtain that $x_i(t)$ is defined on $R_+$ and there is a constant $T_1\geqslant T_0$ such that

$$x_i(t)\leqslant x_i^*(t)+\varepsilon_0,\quad \text{for all } t\geqslant T_1,\ i=1,2,\dots,n.\tag{3.6}$$

Choose a constant $M_i=\sup\{x_i^*(t)+\varepsilon_0\colon\ t\geqslant 0\}$ for each $i=1,2,\dots,n$, then $0<M_i<\infty$ and $M_i$ is independent of any positive solution of system (1.1). Obviously, we have $x_i(t)\leqslant M_i$ for all $t\geqslant T_1$ and $i=1,2,\dots,n$.

Next, we prove that there is a positive constant $m_i$ such that

$$\liminf_{t\to\infty}x_i(t)\geqslant m_i,\quad \text{for each } i=1,2,\dots,n.\tag{3.7}$$

We first prove that

$$\limsup_{t\to\infty}x_i(t)\geqslant\varepsilon_0,\quad \text{for all } i=1,2,\dots,n.\tag{3.8}$$

In fact, if conclusion (3.8) is not true, then there exist an $i\in\{1,2,\dots,n\}$ and $T_2\geqslant T_1$ such that $x_i(t)<\varepsilon_0$ for all $t\geqslant T_2$. Choose a constant $\tau_1>0$ such that

$$\int_{-\infty}^{-\tau_1}k_i(s)\,ds\leqslant\frac{\varepsilon_0}{M},$$

where

$$M = \sup\{x_j(t+s)\colon t \in R_+,\ s \in R_-,\ j = 1, 2, \ldots, n\},$$

then for any $t \geqslant T_2 + \tau_1$ and $t \neq t_k$, we can choose an integer $l \geqslant 0$ such that

$$t = T_2 + \tau_1 + l\omega_i + v,$$

where $v \in [0, \omega_i)$ is a constant. Then we have

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &\geqslant x_i(t)\left(a_i(t) - b_{ii}(t)\varepsilon_0 - \sum_{j=1,\ j\neq i}^{n} b_{ij}(t)\left(M\int_{-\infty}^{-\tau_1} k_i(s)\,\mathrm{d}s + \int_{-\tau_1}^{0} k_i(s)\big(x_j^*(t+s) + \varepsilon_0\big)\,\mathrm{d}s\right)\right)\\
&\geqslant x_i(t)\left(a_i(t) - \sum_{j=1,\ j\neq i}^{n} b_{ij}(t)\mu_{ij}(t) - b_{ii}(t)\varepsilon_0 - 2\sum_{j=1,\ j\neq i}^{n} b_{ij}(t)\varepsilon_0\right).
\end{aligned}$$

Integrating above inequality from $T_2 + \tau_1$ to $t$, from (3.4) and (3.5) we obtain

$$\begin{aligned}
x_i(t) &= x_i(T_2+\tau_1)\exp\left(\int_{T_2+\tau_1}^{t}\left(a_i(s) - b_{ii}(s)x_i(s) - \sum_{j=1,\ j\neq i}^{n} b_{ij}(s)\int_{-\infty}^{0} k_i(v)x_j(s+v)\,\mathrm{d}v\right)\mathrm{d}s + \sum_{T_2+\tau_1\leqslant t_k<t}\ln h_{ik}\right)\\
&\geqslant x_i(T_2+\tau_1)\exp\left(\left(\int_{T_2+\tau_1}^{T_2+\tau_1+\omega_i} + \cdots + \int_{T_2+\tau_1+(l-1)\omega_i}^{T_2+\tau_1+l\omega_i} + \int_{T_2+\tau_1+l\omega_i}^{t}\right)\right.\\
&\quad\times\left(a_i(t) - \sum_{j=1,\ j\neq i}^{n} b_{ij}(t)\mu_{ij}(t) - b_{ii}(t)\varepsilon_0 - 2\sum_{j=1,\ j\neq i}^{n} b_{ij}(t)\varepsilon_0\right)\mathrm{d}s\\
&\quad\left.+\left(\sum_{T_2+\tau_1\leqslant t_k<T_2+\tau_1+\omega_i} + \cdots + \sum_{T_2+\tau_1+(l-1)\omega_i\leqslant t_k<T_2+\tau_1+l\omega_i} + \sum_{T_2+\tau_1+l\omega_i\leqslant t_k<t}\right)\ln h_{ik}\right)\\
&\geqslant x_i(T_2+\tau_1)\exp(l\epsilon_0 - \beta_i\omega_i - H_i),
\end{aligned}$$

where

$$\beta_i = \sup_{t\in R_+}\left\{\big|a_i(t)\big| + \sum_{j=1,\ j\neq i}^{n} b_{ij}(t)\mu_{ij}(t) + b_{ii}(t)\varepsilon_0 + 2\sum_{j=1,\ j\neq i}^{n} b_{ij}(t)\varepsilon_0\right\}. \tag{3.9}$$

Therefore, we have $x_i(t) \to \infty$ as $t \to \infty$, which is a contradiction.

Now, we prove that (3.7) holds. Assume that it is not true, then there is an $i \in \{1, 2, \ldots, n\}$ and a sequence $\{\phi_k\} \subset PCB_+$ of initial functions for system (1.1) such that

$$\liminf_{t\to\infty} x_i(t, \phi_k) < \frac{\varepsilon_0}{k^2}, \quad \text{for all } k = 1, 2, \ldots, \tag{3.10}$$

where $x(t, \phi_k)$ is the solution of system (1.1) satisfying the initial condition $x(s) = \phi_k(s)$ for all $s \in R_-$. From (3.5) we have that

$$e^{-H_j} \leqslant h_{jk} \leqslant e^{H_j}, \quad \text{for all } j = 1, 2, \ldots, n,\ k = 1, 2, \ldots.$$

Hence, we can choose an integer $K > \max\{e^{H_j}\colon j = 1, 2, \ldots, n\}$, such that for any solution $x(t)$ of system (1.1), we have if

$$x_j(t_l) \geqslant \frac{\varepsilon_0}{k}, \quad \text{for some } j = 1, 2, \ldots, n,\ l = 1, 2, \ldots,$$

then

$$x_j\big(t_l^+\big) = h_{jl}x_j(t_l) \geqslant \frac{\varepsilon_0}{k}e^{-H_j} > \frac{\varepsilon_0}{k^2}, \quad \text{for all } k \geqslant K,$$

and if

$$x_j(t_l) \leqslant \frac{\varepsilon_0}{k^2}, \quad \text{for some } j = 1, 2, \ldots, n,\ l = 1, 2, \ldots,$$

then

$$x_j(t_l^+) = h_{jl} x_j(t_l) \leqslant \frac{\varepsilon_0}{k^2} e^{H_j} < \frac{\varepsilon_0}{k}, \quad \text{for all } k \geqslant K.$$

From (3.8), (3.10) and above inequality, we obtain that there exist two time sequences $\{s_q^{(k)}\}$ and $\{t_q^{(k)}\}$ such that for each $k = K+1, K+2, \ldots,$

$$0 < s_1^{(k)} < t_1^{(k)} < s_2^{(k)} < t_2^{(k)} < \cdots < s_q^{(k)} < t_q^{(k)} \cdots,$$

$$s_q^{(k)} \to \infty, \qquad t_q^{(k)} \to \infty \quad \text{as } q \to \infty, \tag{3.11}$$

$$x_i(s_q^{(k)}, \phi_k) \geqslant \frac{\varepsilon_0}{k}, \qquad \frac{\varepsilon_0}{k^2} < x_i(s_q^{(k)+}, \phi_k) \leqslant \frac{\varepsilon_0}{k}, \tag{3.12}$$

$$\frac{\varepsilon_0}{k^2} \leqslant x_i(t_q^{(k)}, \phi_k) < \frac{\varepsilon_0}{k}, \qquad x_i(t_q^{(k)+}, \phi_k) \leqslant \frac{\varepsilon_0}{k^2}, \tag{3.13}$$

$$\frac{\varepsilon_0}{k^2} \leqslant x_i(t, \phi_k) \leqslant \frac{\varepsilon_0}{k} \quad \text{for all } t \in (s_q^{(k)}, t_q^{(k)}). \tag{3.14}$$

Let $M^{(k)} = \sup\{x_j(t+s, \phi_k)\colon\ t \in R_+,\ s \in R_-,\ j = 1, 2, \ldots, n\}$. For each $k = K+1, K+2, \ldots,$ we can choose a constant $\tau_1^{(k)} > 0$ such that

$$\int_{-\infty}^{-\tau_1^{(k)}} k_i(s) x_j(t+s, \phi_k)\, \mathrm{d}s \leqslant M^{(k)} \int_{-\infty}^{-\tau_1^{(k)}} k_i(s)\, \mathrm{d}s < \varepsilon_0. \tag{3.15}$$

By (3.6) for each $k = K+1, \ldots$ there exists a $T_1^{(k)} > T_0$ such that

$$x_j(t, \phi_k) \leqslant x_j^*(t) + \varepsilon_0 \quad \text{for all } t \geqslant T_1^{(k)},\ j = 1, 2, \ldots, n. \tag{3.16}$$

Obviously, by (3.11) there is an $N_1^{(k)} > 0$ such that $s_q^{(k)} > T_1^{(k)} + \tau_1^{(k)}$ for a $q \geqslant N_1^{(k)}$ for each $k = K+1, K+2, \ldots.$ Hence, for any $t \in [s_q^{(k)}, t_q^{(k)}]$ and $t \neq t_l$ $(l = 1, 2, \ldots)$ and $q \geqslant N_1^{(k)}$ by (3.15) and (3.16) we have

$$\begin{aligned}
\frac{\mathrm{d}x_i(t, \phi_k)}{\mathrm{d}t} &\geqslant x_i(t, \phi_k)\Bigg(a_i(t) - b_{ii}(t)\big(x_i^*(t) + \varepsilon_0\big) \\
&\qquad - \sum_{j=1,\, j\neq i}^{n} b_{ij}(t) \int_{-\tau_1^{(k)}}^{0} k_i(s)\big(x_j^*(t+s) + \varepsilon_0\big)\, \mathrm{d}s - \sum_{j=1,\, j\neq i}^{n} b_{ij}(t) \int_{-\infty}^{-\tau_1^{(k)}} k_i(s) x_j(t+s, \phi_k)\, \mathrm{d}s\Bigg) \\
&\geqslant x_i(t, \phi_k)\Bigg(a_i(t) - b_{ii}(t)\big(x_i^*(t) + \varepsilon_0\big) - \sum_{j=1,\, j\neq i}^{n} b_{ij}(t)\mu_{ij}(t) - 2\sum_{j=1,\, j\neq i}^{n} b_{ij}(t)\varepsilon_0\Bigg).
\end{aligned}$$

We can choose an integer $l_q^{(k)}$ such that $t_q^{(k)} = s_q^{(k)} + l_q^{(k)}\omega_i + v_q^{(k)}$, where $v_q^{(k)} \in [0, \omega_i)$. Then we obtain from (3.4), (3.12) and (3.13)

$$\begin{aligned}
\frac{\varepsilon_0}{k^2} &\geqslant x_i(t_q^{(k)+}, \phi_k) \\
&\geqslant x_i(s_q^{(k)}, \phi_k) \exp\Bigg(\int_{s_q^{(k)}}^{t_q^{(k)}} \Bigg(a_i(s) - b_{ii}(s)x_i^*(s) - \sum_{j=1,\, j\neq i}^{n} b_{ij}(s)\mu_{ij}(s) \\
&\qquad - b_{ii}(s)\varepsilon_0 - 2\sum_{j=1,\, j\neq i}^{n} b_{ij}(s)\varepsilon_0\Bigg)\, \mathrm{d}s + \sum_{s_q^{(k)} \leqslant t_k \leqslant t_q^{(k)}} \ln h_{ik}\Bigg) \\
&\geqslant x_i(s_q^{(k)}, \phi_k) \exp\Bigg(\int_{s_q^{(k)} + l_q^{(k)}\omega_i}^{t_q^{(k)}} \Bigg(a_i(s) - \sum_{j=1,\, j\neq i}^{n} b_{ij}(s)\mu_{ij}(s)
\end{aligned}$$

$$-b_{ii}(s)\varepsilon_0 - 2\sum_{j=1,\, j\neq i}^{n} b_{ij}(s)\varepsilon_0\Bigg)\,\mathrm{d}s + \sum_{s_q^{(k)}+l_q^{(k)}\omega_i\leqslant t_k\leqslant t_q^{(k)}} \ln h_{ik} - \int_{s_q^{(k)}}^{t_q^{(k)}} b_{ii}(s)x_i^*(s)\,\mathrm{d}s\Bigg)$$

$$\geqslant \frac{\varepsilon_0}{k}\exp\bigl(-\beta_i\omega_i - 2H_i - \gamma_i\bigl(t_q^{(k)} - s_q^{(k)}\bigr)\bigr), \tag{3.17}$$

where $\beta_i$ is defined by (3.9) and $\gamma_i = \sup\{b_{ii}(s)x_i^*(s):\ s\in R_+\} > 0$. Consequently, we have

$$t_q^{(k)} - s_q^{(k)} \geqslant \frac{\ln k - \beta_i\omega_i - 2H_i}{\gamma_i}, \quad \text{for all } q\geqslant N_1^{(k)},\ k = K+1, K+2, \ldots. \tag{3.18}$$

For any $t > s_q^{(k)}$ and $q\geqslant N_1^{(k)}$ we have

$$\int_{-\infty}^{T_1^{(k)}} k_i(u-t)x_j(u,\phi_k)\,\mathrm{d}u \leqslant M^{(k)}\int_{-\infty}^{T_1^{(k)}-t} k_i(s)\,\mathrm{d}s \tag{3.19}$$

and

$$\int_{T_1^{(k)}}^{s_q^{(k)}} k_i(u-t)x_j(u,\phi_k)\,\mathrm{d}u \leqslant M_j\int_{-\infty}^{s_q^{(k)}-t} k_i(s)\,\mathrm{d}s, \tag{3.20}$$

for all $j = 1,2,\ldots,n$ and $j\neq i$. For each $k = K+1, K+2,\ldots$ by (3.11) there exist an $N_2^{(k)}\geqslant N_1^{(k)}$ and a constant $L>0$ such that

$$M^{(k)}\int_{-\infty}^{T_1^{(k)}-s_q^{(k)}} k_i(s)\,\mathrm{d}s \leqslant \frac{\varepsilon_0}{2}, \quad \text{for all } q\geqslant N_2^{(k)} \tag{3.21}$$

and

$$M_j\int_{-\infty}^{-L} k_i(s)\,\mathrm{d}s \leqslant \frac{\varepsilon_0}{2}, \quad \text{for all } j = 1,2,\ldots,n \text{ and } j\neq i. \tag{3.22}$$

We can choose an integer $r_q^{(k)}\geqslant 0$ such that

$$t_q^{(k)} = s_q^{(k)} + L + r_q^{(k)}\omega_i + w_q^{(k)},$$

where $w_q^{(k)}\in[0,\omega_i)$ is a constant.

By (3.18) there exists a large enough $K_1\geqslant K$ such that

$$r_q^{(k)}\varepsilon_0 - \beta_i\omega_i - 2H_i \geqslant \varepsilon_0, \tag{3.23}$$

for all $k\geqslant K_1$, $q\geqslant N_2^{(k)}$. Hence, for any $k\geqslant K_1$, $q\geqslant N_2^{(k)}$ and $t\in[s_q^{(k)}+L, t_q^{(k)}]$ and $t\neq t_l$ $(l = 1,2,\ldots)$, by (3.14) and (3.19)–(3.22) we have

$$\frac{\mathrm{d}x_i(t,\phi_k)}{\mathrm{d}t} = x_i(t,\phi_k)\Bigg(a_i(t) - b_{ii}(t)x_i(t,\phi_k) - \sum_{j=1,\, j\neq i}^{n} b_{ij}(t)\Bigg(\int_{-\infty}^{T_1^{(k)}} + \int_{T_1^{(k)}}^{s_q^{(k)}} + \int_{s_q^{(k)}}^{t}\Bigg)k_i(u-t)x_j(u,\phi_k)\,\mathrm{d}u\Bigg)$$

$$\geqslant x_i(t,\phi_k)\Bigg(a_i(t) - b_{ii}(t)\frac{\varepsilon_0}{k} - \sum_{j=1,\, j\neq i}^{n} b_{ij}(t)M^{(k)}\int_{-\infty}^{T_1^{(k)}-t} k_i(s)\,\mathrm{d}s$$

$$- \sum_{j=1,\, j\neq i}^{n} b_{ij}(t)M_j\int_{-\infty}^{s_q^{(k)}-t} k_i(s)\,\mathrm{d}s - \sum_{j=1,\, j\neq i}^{n} b_{ij}(t)\int_{s_q^{(k)}}^{t} k_i(u-t)\bigl(x_j^*(u)+\varepsilon_0\bigr)\,\mathrm{d}u\Bigg)$$

$$
\begin{aligned}
&\geqslant x_i(t,\phi_k)\left(a_i(t)-b_{ii}(t)\varepsilon_0-\sum_{j=1,\,j\neq i}^{n} b_{ij}(t)\left(\frac{\varepsilon_0}{2}+\frac{\varepsilon_0}{2}+\varepsilon_0+\mu_{ij}(t)\right)\right)\\
&=x_i(t,\phi_k)\left(a_i(t)-\sum_{j=1,\,j\neq i}^{n} b_{ij}(t)\mu_{ij}(t)-b_{ii}(t)\varepsilon_0-2\sum_{j=1,\,j\neq i}^{n} b_{ij}(t)\varepsilon_0\right).
\end{aligned}
\tag{3.24}
$$

Integrating (3.24) from $s_q^{(k)}+L$ to $t_q^{(k)}$, then by (3.4), (3.12), (3.13) and (3.23) we obtain

$$
\begin{aligned}
\frac{\varepsilon_0}{k^2} &\geqslant x_i\big(t_q^{(k)+},\phi_k\big)\\
&\geqslant x_i\big(s_q^{(k)}+L,\phi_k\big)\exp\Bigg(\Bigg(\int_{s_q^{(k)}+L}^{s_q^{(k)}+L+\omega_i}+\cdots+\int_{s_q^{(k)}+L+(r_q^{(k)}-1)\omega_i}^{s_q^{(k)}+L+r_q^{(k)}\omega_i}\\
&\quad+\int_{s_q^{(k)}+L+r_q^{(k)}\omega_i}^{t_q^{(k)}}\Bigg)\left(a_i(s)-\sum_{j=1,\,j\neq i}^{n} b_{ij}(s)\mu_{ij}(s)-b_{ii}(s)\varepsilon_0-2\sum_{j=1,\,j\neq i}^{n} b_{ij}(s)\varepsilon_0\right)\mathrm{d}s\\
&\quad+\Bigg(\sum_{s_q^{(k)}+L\leqslant t_k<s_q^{(k)}+L+\omega_i}+\cdots+\sum_{s_q^{(k)}+L+(r_q^{(k)}-1)\omega_i\leqslant t_k<s_q^{(k)}+L+r_q^{(k)}\omega_i}+\sum_{s_q^{(k)}+L+r_q^{(k)}\omega_i\leqslant t_k\leqslant t_q^{(k)}}\Bigg)\ln h_{ik}\Bigg)\\
&\geqslant \frac{\varepsilon_0}{k^2}\exp\big(r_q^{(k)}\varepsilon_0-\beta_i\omega_i-2H_i\big)\\
&\geqslant \frac{\varepsilon_0}{k^2}\exp(\varepsilon_0)\\
&> \frac{\varepsilon_0}{k^2},
\end{aligned}
$$

which is contradictory. This contradiction shows that there exists a constant $m_i>0$ such that

$$\liminf_{t\to\infty} x_i(t)\geqslant m_i$$

for any positive solution $x(t)$ of system (1.1), i.e. the conclusion of Theorem 3.1 holds. This completes the proof. □

In the following, we will apply Theorem 3.1 to some special cases. Let

$$\eta_i(t)=1-e^{-a_i^M(t-t_k)}+\sum_{l=1}^{k-1}\big(e^{-a_i^M(t-t_{l+1})}-e^{-a_i^M(t-t_l)}\big)\prod_{j=l+1}^{k} h_{ij}^{-1} \tag{3.25}$$

and

$$\zeta_i(t)=e^{-a_i^M(t-t_1)}\prod_{l=1}^{k} h_{il}^{-1},\quad \text{for } i=1,2,\ldots,n. \tag{3.26}$$

**Corollary 3.1.** *Suppose that system* (1.1) *satisfies the following conditions for each $i=1,2,\ldots,n$:*

(H1) $a_i^L, b_{ii}^L>0$.
(H2) *There is a positive constant $\underline{\eta}_i$ such that* $\liminf_{t\to\infty}\eta_i(t)\geqslant\underline{\eta}_i$.
(H3) $\lim_{t\to\infty}\zeta_i(t)=0$.
(H4) *There exists a positive constant $\omega_i$ such that*

$$\liminf_{t\to\infty}\int_t^{t+\omega_i}\left(a_i(s)-\sum_{j=1,\,j\neq i}^{n}\frac{a_j^M b_{ij}(s)}{b_{jj}^L\underline{\eta}_j}\right)\mathrm{d}s+\sum_{t\leqslant t_k\leqslant t+\omega_i}\ln h_{ik}>0. \tag{3.27}$$

(H5) $h_i(t,\nu)=\sum_{t\leqslant t_k<t+\nu}\ln h_{ik}$ *is bounded for all $t\in R_+$ and $\nu\in[0,\omega_i)$.*

*Then system* (1.1) *is permanent.*

**Proof.** From (H4) we directly have

$$\liminf_{t\to\infty}\left(\int_t^{t+\omega_i} a_i(s)\,ds + \sum_{t\leqslant t_k<t+\omega_i} \ln h_{ik}\right) > 0, \quad \text{for all } i=1,2,\ldots,n.$$

Hence from (H1), (H5) and Lemma 2.1 $x_i^*(t)$ is permanence and globally attractive for system (3.1). By assumptions (H1)–(H3) and Lemma 2.2 we have that

$$\limsup_{t\to\infty} x_i^*(t) \leqslant \frac{a_i^M}{b_{ii}^L \underline{\eta}_i}, \quad \text{for all } i=1,2,\ldots,n. \tag{3.28}$$

From (3.27), there exist a positive constant $\varepsilon$ and $T_1$ such that

$$\int_t^{t+\omega_i}\left(a_i(s) - \sum_{j=1,\, j\neq i}^n b_{ij}(s)\left(\frac{a_j^M}{b_{jj}^L\underline{\eta}_j}+\varepsilon\right)\right)ds + \sum_{t\leqslant t_k\leqslant t+\omega_i}\ln h_{ik} \geqslant \varepsilon, \quad \text{for all } t\geqslant T_1. \tag{3.29}$$

By (3.28), for above $\varepsilon$ there exists a $T_2 \geqslant T_1$ such that

$$x_j^*(t) \leqslant \frac{a_j^M}{b_{jj}^L\underline{\eta}_j} + \frac{\varepsilon}{2}, \quad \text{for all } t\geqslant T_2.$$

Then we can choose a constant $T_3 \geqslant T_2$ such that

$$\bar{v}_j \int_{-s}^{T_2-s} k_i(u)\,ds < \frac{\varepsilon}{2}, \quad \text{for all } s\geqslant T_3,\ j=1,2,\ldots,n,$$

where $\bar{v}_j = \sup\{x_j^*(s)\colon\ s\in R_+\}$. Therefore,

$$\begin{aligned}
\mu_{ij}(s) &= \int_0^s k_i(u-s)x_j^*(u)\,du \\
&= \int_0^{T_2} k_i(u-s)x_j^*(u)\,du + \int_{T_2}^s k_i(u-s)x_j^*(u)\,du \\
&\leqslant \bar{v}_j \int_{-s}^{T_2-s} k_i(u)\,du + \left(\frac{a_j^M}{b_{jj}^L\underline{\eta}_j}+\frac{\varepsilon}{2}\right)\int_{T_2-s}^0 k_i(u)\,du \\
&\leqslant \frac{a_j^M}{b_{jj}^L\underline{\eta}_j} + \varepsilon, \quad \text{for all } s\geqslant T_3.
\end{aligned}$$

Therefore, for all $t \geqslant T_3$ we have

$$\begin{aligned}
&\int_t^{t+\omega_i}\left(a_i(s) - \sum_{j=1,\, j\neq i}^n b_{ij}(s)\mu_{ij}(s)\right)ds + \sum_{t\leqslant t_k<t+\omega_i}\ln h_{ik} \\
&\quad\geqslant \int_t^{t+\omega_i}\left(a_i(s) - \sum_{j=1,\, j\neq i}^n b_{ij}(s)\left(\frac{a_j^M}{b_{jj}^L\underline{\eta}_j}+\varepsilon\right)\right)ds + \sum_{t\leqslant t_k<t+\omega_i}\ln h_{ik} \\
&\quad\geqslant \varepsilon > 0.
\end{aligned}$$

By Theorem 3.1, we have that system (1.1) is permanent. This completes the proof of the corollary. □

When system (1.1) is periodic, then we can assume that there is a positive constant $\omega$ and a positive integer $q$ such that $a_i(t+\omega) = a_i(t)$, $b_{ij}(t+\omega) = b_{ij}(t)$, $t_{k+q} = t_k + \omega$ and $h_{ik+q} = h_{ik}$ for all $t \in R_+$, $k = 1,2,\ldots$ and $i, j = 1,2,\ldots,n$. From Remarks 2.1 and 2.2 we see that the fixed positive solution $x_i^*(t)$ of system (3.1) can be chosen to be the $\omega$-periodic solution of system (3.1). Therefore, as a consequence of Theorem 3.1 we have the following result.

**Corollary 3.2.** *Suppose that system* (1.1) *is $\omega$-periodic and for each $i = 1, 2, \dots, n$,* (H1)–(H3) *hold and*

$$\int_0^{\omega}\left(a_i(s) - \sum_{j=1,\, j\neq i}^{n} \frac{a_j^M b_{ij}(s)}{b_{jj}^L \underline{\eta}_j}\right) ds + \sum_{0\leqslant t_k\leqslant\omega} \ln h_{ik} > 0. \tag{3.30}$$

*Then system* (1.1) *is permanent.*

In fact, if system (1.1) is $\omega$-periodic, it is easy to obtain the condition (3.27) holds by (3.30). Furthermore, function $h_i(t,\nu)$ satisfies

$$\left|h_i(t,\nu)\right| = \left|\sum_{t\leqslant t_k<t+\nu} \ln h_{ik}\right| \leqslant \sum_{k=1}^{q} |\ln h_{ik}|, \quad i = 1, 2, \dots, n,$$

for all $t \in R_+$ and $\nu \in [0,\omega)$. Therefore, from Corollary 3.1 we obtain the conclusion of Corollary 3.2 holds.

**Remark 3.1.** In system (1.1), if

$$\int_{-\infty}^{0} k_i(s)x_j(t+s)\,ds = x_j(t) \quad \text{for } i = 1, 2, \dots, n,\ j \neq i,$$

then we can obtain $\mu_{ij}(t) = x_j^*(t)$. Therefore, in this case Theorem 3.1 is the same as Theorem 3.1 in [15] and our result extends Theorem 3.1 in [15].

**Remark 3.2.** The conditions of Theorem 3.1 are very weak, but it is difficult to verify, since $\mu_{ij}(t)$ is difficult to calculate in many cases. Therefore, in some special cases, we can use Corollary 3.1, the conditions in Corollary 3.1 are more easy to check than Theorem 3.1's.

## 4. Global attractivity

In this section we discuss the global attractivity of system (1.1). We have the following result.

**Theorem 4.1.** *Suppose all the conditions of Theorem* 3.1 *hold and there are constant $c_i > 0$ $(i = 1, 2, \dots, n)$ and nonnegative constant $\alpha$ satisfying*

$$c_i b_{ii}(t) - \sum_{j=1,\, j\neq i}^{n} c_j g_{ji}(t) \geqslant \alpha, \quad \textit{for all } t \geqslant 0 \textit{ and } i = 1, 2, \dots, n, \tag{4.1}$$

*where*

$$g_{ij}(t) = \int_{-\infty}^{0} k_i(s) b_{ij}(t-s)\,ds.$$

*Then system* (1.1) *is globally attractive, that is, any two positive solutions $x(t) = (x_1(t), \dots, x_n(t))$ and $y(t) = (y_1(t), \dots, y_2(t))$ of system* (1.1) *satisfy*

$$\lim_{t\to\infty}\left|x_i(t) - y_i(t)\right| = 0, \quad \textit{for all } i = 1, 2, \dots, n.$$

**Proof.** Let $x(t) = (x_1(t), \dots, x_n(t))$ and $y(t) = (y_1(t), \dots, y_2(t))$ be any two positive solutions of system (1.1). From Theorem 3.1, we can obtain that there are positive constants $r$ and $R$ such that

$$r \leqslant x_i(t), y_i(t) \leqslant R \quad \text{for all } t \geqslant 0,\ i = 1, 2, \dots, n. \tag{4.2}$$

Choose a Lyapunov function as follows:

$$V_1(t) = \sum_{i=1}^{n} c_i \left|\ln x_i(t) - \ln y_i(t)\right|.$$

Since for any impulsive time $t_k$ we have

$$V_1(t_k^+) = \sum_{i=1}^{n} c_i \left| \ln h_{ik} x_i(t) - \ln h_{ik} y_i(t) \right| = V_1(t_k),$$

$V_1(t)$ is continuous for all $t \geqslant 0$. For any $t \in R_+$ and $t \neq t_k$, calculating the right upper Dini derivative of $V_1(t)$, we obtain

$$\begin{aligned} D^+ V_1(t) &= \sum_{i=1}^{n} c_i \operatorname{sgn}\big(x_i(t) - y_i(t)\big) \left( -b_{ii}(t)\big(x_i(t) - y_i(t)\big) + \sum_{j=1,\, j\neq i}^{n} b_{ij}(t) \int_{-\infty}^{0} k_i(s)\big(x_j(t+s) - y_j(t+s)\big)\, \mathrm{d}s \right) \\ &\leqslant -\sum_{i=1}^{n} c_i b_{ii}(t)\left|x_i(t) - y_i(t)\right| + \sum_{i=1}^{n} c_i \sum_{j=1,\, j\neq i}^{n} b_{ij}(t) \int_{-\infty}^{0} k_i(s)\left|x_j(t+s) - y_j(t+s)\right| \mathrm{d}s. \end{aligned} \tag{4.3}$$

We define

$$V_2(t) = \sum_{i=1}^{n} \sum_{j=1,\, j\neq i}^{n} c_i \int_{-\infty}^{0} k_i(s) \int_{t+s}^{t} b_{ij}(u-s)\left|x_j(u) - y_j(u)\right| \mathrm{d}u\, \mathrm{d}s.$$

Obviously, $V_2(t)$ is continuous for all $t \geqslant 0$. For any $t \in R_+$ and $t \neq t_k$, calculating the right upper Dini derivative of $V_2(t)$, we have

$$\begin{aligned} D^+ V_2(t) &= \sum_{i=1}^{n} \sum_{j=1,\, j\neq i}^{n} c_i \int_{-\infty}^{0} k_i(s) b_{ij}(t-s)\left|x_j(t) - y_j(t)\right| \mathrm{d}s \\ &\quad - \sum_{i=1}^{n} \sum_{j=1,\, j\neq i}^{n} c_i \int_{-\infty}^{0} k_i(s) b_{ij}(t)\left|x_j(t+s) - y_j(t+s)\right| \mathrm{d}u\, \mathrm{d}s. \end{aligned} \tag{4.4}$$

Let $V(t) = V_1(t) + V_2(t)$. From (4.3) and (4.4), we have

$$\begin{aligned} D^+ V(t) &\leqslant -\sum_{i=1}^{n} c_i b_{ii}(t)\left|x_i(t) - y_i(t)\right| + \sum_{i=1}^{n} \sum_{j=1,\, j\neq i}^{n} c_i g_{ij}(t)\left|x_j(t) - y_j(t)\right| \\ &= -\sum_{i=1}^{n} \left( c_i b_{ii}(t) - \sum_{j=1,\, j\neq i}^{n} c_j g_{ji}(t) \right) \left|x_i(t) - y_i(t)\right| \\ &\leqslant -\alpha \sum_{i=1}^{n} \left|x_i(t) - y_i(t)\right| \quad \text{for all } t \in R_+ \text{ and } t \neq t_k. \end{aligned}$$

From this, for any $t \geqslant 0$ we further have

$$V(t) + \alpha \int_0^t \sum_{i=1}^{n} \left|x_i(s) - y_i(s)\right| \mathrm{d}s \leqslant V(0).$$

Hence $V(t)$ and $\int_0^t \sum_{i=1}^{n} |x_i(s) - y_i(s)|\, \mathrm{d}s$ are bounded. From Lemma 2.4 and (4.2), we have $\sum_{i=1}^{n} |x_i(t) - y_i(t)|$ is uniformly continuous, then by the Barbalat's lemma we obtain that

$$\lim_{t\to\infty} |x_i(t) - y_i(t)| = 0.$$

This completes the proof of the theorem. □

**Corollary 4.1.** *Suppose all the conditions of Corollary* 3.1 *hold and there are constant $c_i > 0$ $(i = 1, 2, \ldots, n)$ and nonnegative constant $\alpha$ satisfying*

$$c_i b_{ii}^L - \sum_{j=1,\, j\neq i}^{n} c_j g_{ji}(t) \geqslant \alpha, \quad \textit{for all } t \geqslant 0 \textit{ and } i = 1, 2, \ldots, n, \tag{4.5}$$

*where*

$$g_{ij}(t)=\int_{-\infty}^{0} k_i(s)b_{ij}(t-s)\,\mathrm{d}s.$$

*Then system* (1.1) *is globally attractive, that is, any two positive solutions* $x(t)=(x_1(t),\dots,x_n(t))$ *and* $y(t)=(y_1(t),\dots,y_2(t))$ *of system* (1.1) *satisfy*

$$\lim_{t\to\infty}\left|x_i(t)-y_i(t)\right|=0,\quad \textit{for all } i=1,2,\dots,n.$$

**Remark 4.1.** In system (1.1), if

$$\int_{-\infty}^{0} k_i(s)x_j(t+s)\,\mathrm{d}s=x_j(t)\quad \text{for } i=1,2,\dots,n,\ j\neq i,$$

then $g_{ij}(t)=b_{ij}(t)$ and Theorem 4.1 is a special case of Theorem 3.2 in [15]. Therefore, our result extends Theorem 3.2 in [15].

**Remark 4.2.** If system (1.1) is an almost periodic and without impulsive effect system, then Theorem 3.1 is a special case of Theorem 1 in [25]. In [25], Teng proved that for a class of almost periodic Lotka–Volterra type $N$-species competitive systems with infinite delays, if the system is permanent then there exists a global attractive positive almost periodic solution. However, for a class of almost periodic Lotka–Volterra type $N$-species competitive systems with infinite delays and fixed time pulses, the similar result has not been found up until now. Therefore, for almost periodic Lotka–Volterra type $N$-species competitive systems with infinite delays and fixed time pulses, an interesting and important open problem is whether there exists a global attractive positive almost periodic solution if the system is permanent.

## 5. Example

In this section we will give an example to illustrate the conclusions obtained in the above sections. We consider the following periodic two-species Lotka–Volterra competitive system with infinite delays and impulse

$$\begin{aligned}
&\left.\begin{aligned}
\dot{x}_1(t)&=x_1(t)\left(a_1(t)-b_{11}(t)x_1(t)-b_{12}(t)\int_{-\infty}^{0} k_1(s)x_2(t+s)\,\mathrm{d}s\right),\\
\dot{x}_2(t)&=x_2(t)\left(a_2(t)-b_{22}(t)x_1(t)-b_{21}(t)\int_{-\infty}^{0} k_2(s)x_1(t+s)\,\mathrm{d}s\right),
\end{aligned}\right\}\quad t\neq t_k,\\
&\left.\begin{aligned}
x_1\left(t^+\right)&=h_{1k}x_1(t),\\
x_2\left(t^+\right)&=h_{2k}x_2(t),
\end{aligned}\right\}\quad t=t_k,\ k=1,2,\dots.
\end{aligned}\tag{5.1}$$

We take $a_1(t)=4$, $a_2(t)=2$, $b_{11}(t)=1.5$, $b_{12}(t)=0.5$, $b_{22}(t)=1$, $b_{21}(t)=1-|\sin\frac{\pi}{2}t|$, $k_1(s)=e^s$, $k_2(s)=e^{2s}$, $h_{1k}=e^{-1}$, $h_{2k}=e$ and $t_k=k$. Obviously, system (5.1) is periodic with period $\omega=2$. For $q=2$, we have $t_{k+q}=t_k+\omega$, $h_{1k+q}=h_{1k}$ and $h_{2k+q}=h_{2k}$ for all $k=1,2,\dots$.

From (3.25) and (3.26) we can obtain

$$\begin{aligned}
\eta_1(t)&=1-e^{-4(t-k)}+\sum_{i=1}^{k-1}e^{k-i}\left(e^{-4(t-i-1)}-e^{-4(t-i)}\right)\\
&=1+\frac{e^4-e^3}{e^3-1}e^{-4(t-k)}-\frac{e^4-1}{e^3-1}e^{-4t+k+3},\\
\zeta_1(t)&=e^{-4(t-t_1)}e^k=e^{-4(t-t_1)+k},\\
\eta_2(t)&=1-e^{-2(t-k)}+\sum_{i=1}^{k-1}e^{i-k}\left(e^{-2(t-i-1)}-e^{-2(t-i)}\right)\\
&=1-\frac{e^3-e^2}{e^3-1}e^{-2(t-k)}-\frac{e^2-1}{e^3-1}e^{-2t-k+3},
\end{aligned}$$

and

$$\zeta_2(t) = e^{-2(t-t_1)}e^{-k} = e^{-2(t-t_1)-k}.$$

Obviously,

$$\lim_{t\to\infty} \zeta_i(t) = 0, \quad \text{for } i = 1, 2,$$

$$\liminf_{t\to\infty} \eta_1(t) \geqslant \frac{e^4-1}{e(e^3-1)} = \underline{\eta}_1 \quad \text{and} \quad \liminf_{t\to\infty} \eta_2(t) \geqslant \frac{e^2-1}{e^3-1} = \underline{\eta}_2.$$

Since

$$\int_0^\omega \left(a_1(s) - \frac{a_2^M b_{12}(s)}{b_{22}^L \underline{\eta}_2}\right) \mathrm{d}s + \sum_{0\leqslant t_k\leqslant\omega} \ln h_{1k} = 2\left(4 - \frac{e^3-1}{e^2-1}\right) - 2$$
$$\approx 0.0256$$

and

$$\int_0^\omega \left(a_2(s) - \frac{a_1^M b_{21}(s)}{b_{11}^L \underline{\eta}_1}\right) \mathrm{d}s + \sum_{0\leqslant t_k\leqslant\omega} \ln h_{2k} = \int_0^2 \left(2 - \frac{4e(e^3-1)}{1.5(e^4-1)}\left(1 - \left|\sin\frac{\pi}{2}s\right|\right)\right) \mathrm{d}s + 2$$
$$\approx 4.1241,$$

we obtain all conditions in Corollary 3.2 for system (5.1) hold. Therefore, from Corollary 3.2 we see that system (5.1) is permanent.

For any constants $c_1 > 0$ and $c_2 > 0$, from

$$c_1 b_{11}(t) - c_2 \int_{-\infty}^0 k_2(s) b_{21}(t-s)\,\mathrm{d}s = 1.5c_1 - c_2 \int_{-\infty}^0 e^{2s}\left(1 - \left|\sin\frac{\pi}{2}(t-s)\right|\right) \mathrm{d}s$$
$$\geqslant 1.5c_1 - c_2$$

and

$$c_2 b_{22}(t) - c_1 \int_{-\infty}^0 k_1(s) b_{12}(t-s)\,\mathrm{d}s = c_2 - c_1 \int_{-\infty}^0 0.5e^s\,\mathrm{d}s$$
$$= c_2 - 0.5c_1$$

for all $t \in R_+$, we can choose $c_1 = c_2 = 1$. Therefore, condition (4.5) in Theorem 4.1 holds. Therefore, from Theorem 4.1 system (5.1) is globally attractive.

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